ON SMALL NON-UNIFORM HYPERGRAPHS WITHOUT PROPERTY B

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For a given hypergraph \( H = (V, E) \) consider the sum \( q(H) \) of \( 2^{-|e|} \) over \( e \in E \). Consider the class of hypergraphs whose smallest edge is of size \( n \) and for which every 2-colouring has a monochromatic edge. Let \( q(n) \) be the smallest value of \( q(H) \) in this class.

We provide a survey of the known bounds on \( q(n) \) and make some minor refinements.

1. Introduction. A hypergraph \( H = (V, E) \) is a finite set of vertices \( V \) and a set of edges \( E \) where each edge is a set of at least two vertices. A 2-colouring of \( H \) is an assignment of colour blue or red to each vertex in \( H \). A 2-colouring is proper if each edge in \( H \) is not monochromatic. We say that \( H \) is 2-colourable if it admits a proper 2-colouring. A hypergraph is said to be \( n \)-uniform if all of its edges have cardinality \( n \). So a graph is just a 2-uniform hypergraph.

A famous Erdős–Hajnal problem is find the minimum number of edges \( m(n) \) in an \( n \)-uniform hypergraph that is not 2-colourable. The best known asymptotic bounds are

\[
c \sqrt{\frac{n}{\ln n}} 2^n \leq m(n) \leq (1 + o(1)) \frac{e \ln 2}{4} n^2 2^n,
\]

for a positive constant \( c \). The lower bound was proved by Radhakrishnan and Srinivasan [10] and then another proof was given by Cherkashin and Kozik [4]; the upper bound is due to Erdős [7] and stays without improvements from 1963. A survey [12] is devoted to this problem and related topics.

Now let us pass to a non-uniform case. For a given hypergraph \( H = (V, E) \) define the quantity

\[
q(H) := \sum_{e \in E} 2^{-|e|}.
\]

Note that \( q(H) \) is the expectation of red edges in a random red-blue colouring of \( V \) in which vertices get red colour with probability 1/2 independently on each other, and thus \( q(H) \) is twice smaller than the expectation of monochromatic edges in such a colouring.

Erdős [6] asked in 1963 whether the function \( q(n) \) is unbounded, where \( q(n) \) is the minimal value of \( q(H) \) over non-2-colourable hypergraphs \( H \) with the minimal size of

*The work is supported by the Ministry of Education and Science of Bulgaria, Scientific Programme “Enhancing the Research Capacity in Mathematical Sciences (PIKOM)”, No. DO1-67/05.05.2022

2020 Mathematics Subject Classification: 05C15, 05C65.

Key words: non-uniform hypergraphs, hypergraph colouring, property B.
edge $n$. Beck [3] in 1978 proved that $q(n) \geq c \log^* t$ for $\log^*$ being the iterated logarithm and some positive constant $c$.

Then in 2008 Lu [8] announced a proof of a bound $q(n) \geq c \frac{\ln n}{\ln \ln n}$ but it turned out to work only for simple hypergraphs (a hypergraph is simple if every pair of edges shares at most 1 vertex). Shabanov [15] improved the lower bound for the class of simple hypergraphs to $c\sqrt{n}$ (he also made some refinements for hypergraphs with girth bounded from below in [14]). The best known asymptotic bounds on $q(n)$

$$c \ln n \leq q(n) \leq (1 + o(1)) \frac{e \ln 2}{4} n^2,$$

where the lower bound is proved by Duraj, Gutowski, and Kozik [5] and the upper bound is a direct consequence of the Erdős upper bound on $m(n)$ and a straightforward estimate $q(n) \leq m(n) \cdot 2^{-n}$.

The main contribution of this note is a twice better asymptotic upper bound on $q(n)$.

**Theorem 1.1.** Let $n$ be an integer. Then

$$q(n) \leq (1 + o(1)) \frac{e \ln 2}{8} n^2.$$

The proof is based on (probabilistic) alteration method, the result being a random hypergraph which is the union of an $n$-uniform hypergraph and an $[n^2/4]$-uniform hypergraph. Akhmejanova [2] refined some lower bounds on $q(n)$ for hypergraphs with only two different edge cardinalities. Radhakrishnan and Srinivasan [11] refined some lower bounds on $q(n)$ for the class of hypergraphs which locally have edges of comparable size.

**2. The case of small $n$.** The values of $m(n)$ are known only for $n \leq 4$. We have $m(2) = 3$ with the only example of a triangle graph; and $m(3) = 7$ with the only example of the Fano plane. The best known upper bounds for small $n$ are reached by explicit examples; the current situation is outlined in Aglave, Amarnath, Shannigrahi and Singh [1]. No example of $q(n) < 2^{-n}m(n)$ is known. In this section we present an example of a hypergraph $H$ with the smallest edge with 4 elements such that

$$2^{-4}m(4) < q(H) < 2^{-4}(m(4) + 1),$$

and the structure of $H$ completely differs from the known examples for $m(4)$.

Now focus on $n = 4$. In this case Seymour [13] and Toft [16] independently showed that $m(4) \leq 23$. They used the example of a hypergraph on 11 vertices with the following edges:

$$\{1, 2, 9, 10\}, \{3, 4, 9, 10\}, \{5, 6, 9, 10\}, \{7, 8, 9, 10\},$$

$$\{1, 2, 9, 11\}, \{3, 4, 9, 11\}, \{5, 6, 9, 11\}, \{7, 8, 9, 11\},$$

$$\{1, 2, 10, 11\}, \{3, 4, 10, 11\}, \{5, 6, 10, 11\}, \{7, 8, 10, 11\},$$

$$\{1, 3, 5, 8\}, \{1, 3, 6, 7\}, \{1, 4, 5, 7\}, \{1, 4, 6, 7\}, \{1, 4, 6, 8\},$$

$$\{2, 3, 5, 7\}, \{2, 3, 6, 7\}, \{2, 3, 6, 8\}, \{2, 4, 5, 7\}, \{2, 4, 5, 8\}, \{2, 4, 6, 8\}.$$

Östergård [9] shows by a complicated computer search that $m(4) = 23$, and there is only one example on at most 11 vertices.

We provide an example of a hypergraph $H$ with sixteen vertices, twenty 4-edges and
Fig. 1. An explicit definition of $H_4$. Points are vertices and lines are edges. In this picture, opposite white points are the same. For example, the vertical line appears to contain 5 points, but the 2 white points are the same, so it just contains 4 points.

Sixty 8-edges which is not 2-colourable. It means that $q(H) = \frac{95}{64} = \frac{23}{16} + \frac{3}{64} < \frac{24}{16}$ which means that this is better than any known 4-graph, except shows the Seymour–Toft graph.

By construction $H$ consists of a 4-uniform part $H_4$ and an 8-uniform part $H_8$. The 4-uniform part $H_4$ is an affine plane over $GF(4)$. It may be also defined in an explicit way, see Fig. 1 (given as an example in a discussion in math.stackexchange.com by the user Matt). A direct computation shows that $H_4$ has 120 proper 2-colourings and every proper 2-colouring has exactly 8 red and 8 blue vertices. Thus these colourings form 60 “opposite” pairs, i.e. colourings in a pair can be obtained from each other by swapping the colours. Then taking red vertices of one member from each pair as an 8-edge one gets $|E(H_8)| = 60$ and $H_4 \cup H_8$ has no proper 2-colouring as desired.

3. Proof of Theorem 1.1. To avoid rounding in calculations, assume that $n$ is even; the case of odd $n$ is analogous. Consider a set of vertices $V$ with the cardinality $v = \frac{n^2}{2}$, and choose $m$ random edges uniformly and independently; the number $m$ will be specified later. Fix a colouring $C$; clearly the probability of the event that a randomly chosen edge is monochromatic is equal to

$$p := \frac{v_1}{m} + \frac{v_2}{m},$$

where $v_1$ and $v_2$ denote the numbers of vertices of the first and second colour, respectively. Hence, since the edges are chosen independently, the probability that after choosing $m$

\footnote{K. Vorob'ev mentioned that the blue sets of those colourings form a 3-(16,8,12) design.}
random independent edges the colouring $C$ is proper is $(1-p)^m$. Let

$$q := \frac{2\binom{n/2}{m}}{\binom{n}{m}}.$$  

It is well-known that

$$q = (1 + o(1)) \frac{2}{e \cdot 2^n}.$$  

Note that $p \geq q$ because of the convexity of the sequence $\left\{ \binom{t}{n} \right\}_{t \geq 0}$. Since the total number of colourings is $2^{n^2/2}$, and the probability that a fixed colouring is proper is bounded by $(1-q)^m$, the expectation of the number of proper colouring is at most

$$2^{n^2/2} (1-q)^m < e^{\ln 2 \cdot n^2/2 - q m}.$$  

(We use a standard inequality $1 - t < e^{-t}$, $t > 0$.)

To get the Erdős upper bound one should take

$$m = (1+o(1)) \frac{\ln 2}{4} n^2 2^n$$

and check that such a choice leads to $\ln 2 \cdot n^2/2 - q m < 0$ which means that with a positive probability a random graph with $m$ edges has no proper 2-colouring.

For our purpose we need a twice smaller number of edges, i.e. $m' = m/2$; then the expectation of the number of proper colouring is at most $2^{n^2/4}$, so with a positive probability a random graph $H_1$ with $m'$ edges of size $n$ has at most $2^{n^2/4}$ proper colourings. For each such a colouring $C$ we consider an edge $e_C$ of size $n^2/4$ which is monochromatic in $C$. Let $H_2$ be a hypergraph on the same vertex set $V$ consisting of all such edges $e_C$, and let $H$ be the union of $H_1$ and $H_2$. Then

$$q(H) = q(H_1) + q(H_2) \leq (1+o(1)) \frac{e \ln 2}{8} n^2 + 1 = (1+o(1)) \frac{e \ln 2}{8} n^2,$$

as desired.

**Acknowledgments.** I am grateful to Alexey Gordeev for an independent computer search and to Konstantin Vorob’ev and Gábor Damásdi for their helpful for his remarks. My final thanks go to the reviewers and the editorial staff.

**REFERENCES**


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МАЛКИ НЕРАВНОМЕРНИ ХИПЕРГРАФИ БЕЗ СВОЙСТВО В

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За даден хиперграф $H = (V, E)$ нека да разгледаме сумата $q(H)$ на $2^{-|e|}$ върху $e \in E$. Разглеждаме класа хиперграфи, чието най-малко ребро е с тегло $n$ и така да не съществува $2$-цветно негово ребро. Нека $q(n)$ е най-малката стойност на $q(H)$ в този клас. Представяме преглед на известните граници на $q(n)$ и правим някои малки подобрения.

Ключови думи: неравномерни хиперграфи, оцветяване на хиперграфи, свойство B.