SORTING IN CONSTRUCTION OF RESOLUTIONS OF COMBINATORIAL DESIGNS

Antoaneta Tsvetanova¹, Stela Zhelezova²

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
Sofia, Bulgaria

e-mails: ¹a.cvetanova@math.bas.bg, ²stela@math.bas.bg

There are different kinds of sorting algorithms. Each algorithm has its own advantages and disadvantages depending on the data they process. We are interested in the performance of sorting algorithms in the case of construction of resolutions of combinatorial designs. Studying their performance for this special class of problems will give us the opportunity to improve, if possible, the speed of our software for solving similar problems. We use C++ and, for part of our investigations, the computer algebra system GAP.

Keywords: sorting, classification algorithm, combinatorial design, design resolution.

СОРТИРАНЕ НА ДАННИ ПРИ КОНСТРУИРАНЕ НА РЕЗОЛЮЦИИ НА КОМБИНАТОРНИ ДИЗАЙНИ

Антоанета Цветанова¹, Стела Железова²

Институт по математика и информатика, Българска академия на науките,
София, България

e-mails: ¹a.cvetanova@math.bas.bg, ²stela@math.bas.bg

Съществуват много и различни алгоритми за сортиране. Всеки алгоритъм има свои предимства и недостатъци в зависимост от данните, които обработва. Разглеждаме работата на някои алгоритми за сортиране при задачи за конструиране на резолюции на комбинаторни дизайни. Проучването на тяхното представяне, за този специален клас проблеми, ще ни даде възможност да подобрим, ако е възможно, скоростта на нашия софтуер за конструиране на резолюции на комбинаторни дизайни. Използваме C++ и също така, за част от нашите изследвания, системата за компютърна алгебра GAP.

Ключови думи: сортиране, класификационен алгоритъм, резолюция на комбинаторен дизайн.

* https://doi.org/10.55630/mem.2024.53.057-064
Partially supported by the Bulgarian National Science Fund under Contract No KP-06-H62/2-2022.

2020 Mathematics Subject Classification: 68P10, 05-08, 05B25, 05B07.
1. Introduction. This paper presents research on the applicability of a number of sorting techniques in the classification of resolutions of combinatorial designs. For the basic knowledge on resolutions of combinatorial designs refer, for instance, to [11].

Let $V = \{P_i\}_{i=1}^v$ be a finite set of points, and $B = \{B_j\}_{j=1}^b$ — a finite collection of $k$-element subsets of $V$, called blocks. If any 2-subset of $V$ is contained in exactly $\lambda$ blocks of $B$, then $D = (V, B)$ is a 2-$(v, k, \lambda)$ design. Each point is in $r$ blocks.

A parallel class is a partition of set of points by blocks. There are $q$ blocks in a parallel class. A resolution of the design is a partition of the collection of blocks into parallel classes. The design is resolvable if it has at least one resolution. Two resolutions are isomorphic if there exists an automorphism of the design transforming each parallel class of the first resolution into a parallel class of the second one. An automorphism of a resolution is an automorphism of the underlying design which maps each parallel class to a parallel class of the same resolution. The set of all resolution automorphisms is a group.

Resolutions of combinatorial designs are used in different types of error-correcting codes [10, 12], in network coding [14], cryptography [16] etc. Part of the research on the topic has been done by computational methods. Many interesting classification results are obtained by computer, for instance, in [1, 3, 20, 21, 23]. Some authors use classification algorithms based on canonical augmentation [2] and also algorithms for computing of the code weight distribution in which no sorting is needed [17]. Equally important the orderly generation algorithms are in use too [6, 7, 13, 26]. They are based on the backtrack search technique and thus are with exponential time complexity. Nevertheless, for relatively small parameter sets and in combination with specific parameter depending theoretical conditions, they can be very successful. Apart from the use of theoretical restrictions, the algorithm implementation can be improved by refining the computation too.

In this paper we are interested in the performance of an auxiliary algorithm, namely the involved sorting algorithm. It appears in an orderly generation algorithm because it is connected with a particular lexicographic order imposed on the constructed objects. Sorted objects facilitate search and comparison and improve data operations efficiency. This, together with the frequently great number of combinatorial objects which are considered, leads to the impact of time spent for sorting on the whole computation time.

There are different kinds of sorting algorithms. Their performance and parameters are studied in depth by Knuth [15]. In our research we do not focus on their time complexity. It is known for all of them. Also there are papers on comparative analysis of various sorting algorithms in the general case [25, 30] and in case of data with a given specific distribution [4, 22]. We are interested in the performance of comparison-based unstable sorting algorithms suitable for the classification problems we consider. We study one and two-dimensional sorting which appears in the construction of resolutions of combinatorial designs. For each of them we apply a Cocktail Sort algorithm [15], the Introsort algorithm [18] (the built-in function in C++ [24]) and the Pattern-defeating quicksort algorithm [19] (the built-in GAP function [8]). For each sorting algorithm the maximal, minimal and the average time for execution of the sorting tasks are measured.

2. Sorting problems. We consider the construction of resolutions of a given combinatorial design with a prescribed automorphism group of order $i$ ($G_i$). The generation method supposes the employment of a chosen lexicographic order. The elements involved
in sorting are points or/and blocks of a particular combinatorial design. We can always denote them by integers. Design points can be enumerated \( V = \{1, 2, \ldots, v\} \). Point labels are used to define a lexicographic order on the blocks and next to assign numbers to the blocks according to this order \( B = \{1, 2, \ldots, b\} \). The place and the size of a sorting task depends on the particular problem we consider.

2.1. Problem 1. In [27] parallelisms of \( PG(3, 7) \) invariant under an automorphism group \( (G_{49}) \) of order 49 with some additional properties are constructed. They comprise resolutions of the point-line 2-(400, 8, 1) design. It has \( b = 2850 \) blocks, each parallel class has \( q = 50 \) blocks and a resolution has \( r = 57 \) parallel classes. The design is known and the resolution construction starts with already lexicographically ordered design blocks. One way to derive resolutions invariant under \( G_{49} \) is by construction of all possibilities for parallel classes firstly. This can be done by backtrack search on the block orbits under \( G_{49} \). These block orbits are considered on the lexicographically ordered design blocks. Therefore the first block in each orbit is the smallest one but the other blocks appear depending on the action of the considered automorphism group. Thus the blocks in each orbit are not necessarily in lexicographic order. In this case a parallel class consists of entire block orbits under \( G_7 \) – a particular subgroup of \( G_{49} \) and hence its blocks cannot appear in the chosen lexicographic order by construction. The structure of a parallel class under the given above conditions is presented in the first row of Table 1. Here \( f \) stands for a fixed under a particular \( G_7 \) block while \( O_{ij} \) denotes \( j \)-th design block from the \( i \)-th orbit of length 7 under \( G_7 \). The second row corresponds to a particular parallel class with this structure. Each block is represented by its consecutive number in the chosen lexicographic order. For the purpose of our research we work with 3000 parallel classes. Each parallel class can be stored in a proper one-dimensional data container of integers named \( \text{Par}_\text{class} \) with size \( q + 1 = 51 \).

Table 1. A parallel class of a point-line 2-(400, 8, 1) design fixed by a \( G_7 \)

| \( f_1 \) | \( f_2 \) | \( f_3 \) | \( f_4 \) | \( f_5 \) | \( f_6 \) | \( f_7 \) | \( O_1^5 \) | \( O_2^5 \) | \( O_3^5 \) | \( O_4^5 \) | \( O_5^5 \) | \( O_6^5 \) | \( O_7^5 \) | \( O_1^4 \) | \( O_2^4 \) | \( O_3^4 \) | \( O_4^4 \) | \( O_5^4 \) | \( O_6^4 \) | \( O_7^4 \) |
| 9 | 58 | 142 | 212 | 240 | 331 | 359 | 429 | 716 | 1066 | 1408 | 1747 | 2093 | 2432 | 2778 | 779 | 1125 | 1471 | 1810 | 2156 | 2495 | 2841 |

2.2. Problem 2. In [28] the classification of point-cyclic resolutions of cyclic 2-(45, 3, 1) designs is considered. A design is cyclic if it has a cyclic automorphism group of order \( v \). A resolution with an automorphism permuting its points in one cycle is called point-cyclic. There are 11616 cyclic designs with these parameters [5]. It is well known that \( G_{45} \) partition the design blocks in one short block orbit of length \( v/3 = 15 \) and 7 block orbits of length \( v = 45 \). For tightness of the stored data such a design can be presented only by one block from each orbit (the smallest one in the chosen lexicographic order). In our case 8 blocks are enough. Each design block is a triple of points. With the agreement that the 8 block orbit representatives start with the first point they can be written by the numbers of the other two points only. The designs are stored in a file where each cyclic design corresponds to a row. The second row of Table 2 shows one such record with 8 groups of numbers. At the third row the corresponding design blocks are written. The first block represents the short orbit while the next 7 blocks are representatives of the long block orbits. We want to construct all point-cyclic resolution of this cyclic design.

<table>
<thead>
<tr>
<th>( F_1 )</th>
<th>( F_2 )</th>
<th>( F_3 )</th>
<th>( F_4 )</th>
<th>( F_5 )</th>
<th>( F_6 )</th>
<th>( F_7 )</th>
<th>( F_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>58</td>
<td>142</td>
<td>212</td>
<td>240</td>
<td>331</td>
<td>359</td>
<td>429</td>
</tr>
<tr>
<td>716</td>
<td>1066</td>
<td>1408</td>
<td>1747</td>
<td>2093</td>
<td>2432</td>
<td>2778</td>
<td>779</td>
</tr>
<tr>
<td>1125</td>
<td>1471</td>
<td>1810</td>
<td>2156</td>
<td>2495</td>
<td>2841</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
One approach is to start with getting all design blocks by $G_{45}$. As can be seen from the last block in each orbit (the last row of Table 2) a cyclic shift of the block points leads to blocks with unordered points. The other problem is that if we write the orbits one after another to obtain all 330 blocks of a considered cyclic 2-(45, 3, 1) design they will appear not in the chosen lexicographic order. Therefore we need to perform two-dimensional sorting on all design blocks. They can be stored in a proper two-dimensional data container of integers named Design blocks with size $b + 1 = 331$ and $k + 1 = 4$.

<table>
<thead>
<tr>
<th>Orbit Length</th>
<th>Short(15)</th>
<th>Long(45)</th>
<th>Long</th>
<th>Long</th>
<th>Long</th>
<th>Long</th>
<th>Long</th>
</tr>
</thead>
<tbody>
<tr>
<td>a record</td>
<td>16, 31</td>
<td>2, 4</td>
<td>5, 13</td>
<td>8, 22</td>
<td>7, 29</td>
<td>11, 27</td>
<td>10, 33</td>
</tr>
<tr>
<td>Block orbits</td>
<td>1, 16, 31</td>
<td>1, 2, 4</td>
<td>1, 5, 13</td>
<td>1, 8, 22</td>
<td>1, 7, 29</td>
<td>1, 11, 27</td>
<td>1, 10, 35</td>
</tr>
<tr>
<td>2nd block</td>
<td>2, 17, 32</td>
<td>2, 3, 5</td>
<td>2, 6, 14</td>
<td>2, 9, 23</td>
<td>2, 8, 30</td>
<td>2, 12, 28</td>
<td>2, 11, 36</td>
</tr>
<tr>
<td>Last block</td>
<td>15, 30, 45</td>
<td>45, 1, 3</td>
<td>45, 4, 12</td>
<td>45, 7, 21</td>
<td>45, 8, 30</td>
<td>45, 10, 26</td>
<td>45, 9, 34</td>
</tr>
</tbody>
</table>

### 3. Sorting algorithms.

Our programs are written in C++. Microsoft Visual C++ 2022 is used and the results are obtained on on Intel i3-7100U @ 2x2.4 GHz. The data we have to sort comprises only integers. Its entire size is fixed by the design parameters and hence is known in advance. The records in which they are included are not so long. The size of a Problem 1 like task is determined by $q$ and usually is less than one hundred. For a Problem 2 like task the size is limited by $r$ or by $b$ and for the parameter sets we consider does not exceed one thousand. But the number of considered objects typically is very big. For instance there are 14227090 possible parallel classes invariant under a particular automorphism group of order 8 of the point-line 2-(156, 6, 1) design (Problem 1 like task) and there are 2353310 cyclic 2-(57, 3, 1) designs (Problem 2 like task). These are the reasons to choose as suitable for our investigations a Cocktail Sort algorithm, the Introsort algorithm and one of the sorting algorithms included in GAP.

An orderly algorithm for construction of design resolutions generally includes one and two-dimensional sorting therefore it will be more effective if the data is stored in one and the same type containers for both problems.

#### 3.1. Cocktail Sort.
The Cocktail Sort algorithm (CSA) is based on the Bubble Sort algorithm (BSA). A BSA passes through an array from the beginning (left) to the end (right), compares elements and swaps them if their order is not correct [15, 24]. In the first iteration the largest element is settled on its exact place. Next the second largest element is moved to the right up to find its place etc. until all data is sorted. CSA does the same but in both directions. The first iteration is the same as for BSA while the second starts from the element before the just sorted one and goes from right to the left such that the smallest element is settled on place. If $N$ is the number of sorted elements in the best case, the CSA time complexity is $O(N)$ and in the worst case is $O(N^2)$ as is for BSA. Nevertheless, there is an empirical analysis of CSA which shows that it is less than two times faster than BSA [29].

We implement CSA for Problem 1 in a standard manner in the function CocktailSort (int arg[], int right). An one-dimensional array of integers int Par_class[q+1]; is used to store each parallel class in Problem 1. It is passed to the function as a parameter. Its length (right) also has to be passed because the one-dimensional sort function is used for different data in many places in the implementation of our construc-
tion algorithm. For Problem 1 the function is called in the following way: CocktailSort(Par_class, q);

For Problem 2 design blocks are stored in int Deign_blocks[b+1][k+1] – a standard two-dimensional array of integers. It is passed to the function as parameter but further CSA needs one more parameter. For a particular sorting task regarding design parameters it is known in how many positions two rows of the two-dimensional data array differ. The parameter dept depends on the number of these positions. The function declaration is CocktailSort(int arg[] [k+1], int right, int dept). Before swapping two unlike elements in position dept of rows i and i + 1 we have to check if they coincide in all positions before dept. Thus for Problem 2 the function is called as:

for (int i = 1; i < k; i++)
    CocktailSort(Deign_blocks[] [k+1], b, i);

3.2. Introsort. Introsort (Introspective sort) is a hybrid sorting algorithm incorporating Insertion sort for small set lengths, Heapsort for a recursion depth greater than a particular level and Quicksort in all other cases. More about the algorithm can be found in [18]. This combination ensures good time complexity in the worst and in the average case $O(N \log N)$, where $N$ is the number of sorted elements.

In our algorithm implementation we use the Introsort algorithm incorporated in the C++ Standard Library as a built-in std::sort() function. It is included in the <algorithm> header file of C++. The sort() function cannot be used with a fixed two-dimensional array because an array cannot be assigned to another array. Therefore to apply the Introsort algorithm by the sort() function the data should be stored in std::array or std::vector classes. The main difference between them is that the first class is a fixed-size container while the second is a dynamic one. We use vectors but since the size of the data is known in advance the dynamic properties of the vectors are not employed here. This is important because a vector growth can lead to growth of the processing time.

Each parallel class in Problem 1 is stored in a vector vector<int> Par_class(q+1); and the design blocks in Problem 2 are written in a vector of vectors vector<vector<int>> Deign_blocks(b+1, vector<int>(k+1));. The range of sorting is given by iterators to the begin and to the end of the corresponding vector. We do not specify the function’s third parameter as ascending order is used by default and it is suitable for the considered objects. Thus for Problem 1 we have sort(Par_class.begin(), Par_class.end()); and for Problem 2 sort(Deign_blocks.begin(), Deign_blocks.end());.

3.3. Sorting data in GAP. The system for computational discrete algebra (GAP) is a freely distributed software. It is especially useful in group computations. Generally we use GAP to obtain the required for the constructive algorithm automorphism groups and their subgroups. In this paper we employ GAP in the needed sorting task. Three different sorting algorithms are involved in GAP. We apply the default one which is implemented in GAP’s function Sort(). It comprises the Pattern-defeating quicksort algorithm [19]. This quicksort modification is a hybrid sorting algorithm which contains the same sorting algorithms as Introsort, but with a different combining strategy. A novel scheme for partitioning in the quicksort part of the algorithm and a concept of a bad partitioning instead of exploiting a recursion depth are among the divergences from the Introsort algorithm. For a few common patterns this sorting algorithm achieves linear
time complexity but in all other cases its time complexity is like that of the Introsort algorithm.

The syntax of the function is \texttt{Sort(list,func)} [9]. The parameter \texttt{func} gives the rule for comparing the elements and is omitted if the regular ascending order is adopted. Therefore to do necessary sort tasks in \texttt{GAP} we have to store our data in data structure called \texttt{list}. Its elements are written between square brackets \texttt{[ , ]} and separated by commas. For Problem 1 the sort function is called as \texttt{Sort(Par_class);} For Problem 2 the data is stored in list of lists. We need to define a specific sorting function which answers how to compare two sub-lists that coincide in the first position. In this case the sort function is called as follows:

\begin{verbatim}
Sort(Deign_blocks, function(v, w)
\end{verbatim}

4. Comparison. The times taken by the execution of each of the considered sorting algorithms for the problems mentioned above are given in Table 3. For each of them the minimal, maximal and average time for sorting an object is presented. To be more precise we get the running time in five identical trials. The values in the table are the averages of these trials.

It looks like the Introsort algorithm implemented in \texttt{C++} standard library has the best performance in the case of the one-dimensional Problem 1. Our Cocktail sort algorithm implementation is almost two times slower. The data size here is relatively small thus this result can be expected. The minimal time for one-dimensional sorting is the best for the Cocktail sort algorithm implementation because it refers to a case of almost sorted object.

For the two-dimensional Problem 2 the implementation of the Pattern-defeating quicksort algorithm in \texttt{GAP} is with the best performance. In this case our Cocktail sort algorithm implementation is almost 13 times slower. In Problem 2 we have to process 11616 objects so this difference is very important.

Our investigation shows that the implementation of the main constructive algorithm should better use the sort function from the \texttt{C++} standard library. Thus the standard \texttt{C++} arrays have to be replaced by vectors. Another possibility is to use the Pattern-defeating quicksort algorithm implementation for \texttt{C++} which is under a zlib license and can be found at https://github.com/orlp/pdqsort.

\begin{table}[h!]
\centering
\caption{Times taken by the considered algorithm implementations in microseconds}
\begin{tabular}{|c|c|c|c|c|}
\hline
Task & Algorithm & min & max & average \\
\hline
Problem 1 & Cocktail sort & 1 & 30999.7 & 50.4 \\
& \texttt{std::sort()} & 6.4 & 7700.4 & 27.3 \\
& GAP Sort() & 12.4 & 670 & 37.8 \\
\hline
Problem 2 & Cocktail sort & 8844.4 & 186013.6 & 13307.5 \\
& \texttt{std::sort()} & 1588.6 & 115924.8 & 3092.2 \\
& GAP Sort() & 643.3 & 51162.7 & 1064.7 \\
\hline
\end{tabular}
\end{table}
REFERENCES

(IJCIET), 8, no. 1 (2017), 955–978.


11 (2023), 2458.

Section of SIAM, BGSIAM’23 Extended abstracts (2023), 48–49.

cessed 21 December 2023.