ON THE DISTRIBUTION OF $\alpha p^2 + \beta$ MODULO ONE FOR PRIMES $p$ SUCH THAT $p + 2$ HAS NO MORE TWO PRIME DIVISORS

Tatyana L. Todorova
Faculty of Mathematics and Informatics, Sofia University “St. Kliment Ohridski”, Sofia, Bulgaria
e-mail: tlt@fmi.uni-sofia.bg

A classical problem in analytic number theory is to study the distribution of fractional part $\alpha p^k + \beta$, $k \geq 1$ modulo 1, where $\alpha$ is irrational and $p$ runs over the set of primes. We consider the subsequence generated by the primes $p$ such that $p + 2$ is an almost-prime (the existence of infinitely many such $p$ is another topical result in prime number theory) and prove that its distribution has a similar property.

Keywords: linear sieve, almost primes, distribution modulo one.

This work was supported Sofia University Scientific Fund, grant 80-10-99/2023.

2020 Mathematics Subject Classification: 11J71, 11N36.
1. Introduction and statements of the result. The famous prime twins conjecture states that there exist infinitely many primes $p$ such that $p + 2$ is a prime too. This hypothesis is still unproved but in 1973 Chen [2] proved that there are infinitely many primes $p$ for which $p + 2 = P_2$. (As usual $P_r$ denotes an integer with no more than $r$ prime factors, counted according to multiplicity).

Let $\alpha$ be irrational real number and $\|x\|$ denote the distance from $x$ to the nearest integer. The distribution of fractional parts of the sequence $\alpha n^k$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ was first considered by Hardy, Littlewood [5] and Weyl [19]. The problem of distribution of the fractional parts of $\alpha p^k$, where $p$ denotes a prime, first was considered by Vinogradov (see Chapter 11 of [17] for the case $k = 1$, [18] for $k \geq 2$), who showed that for any real $\beta$ there are infinitely many primes $p$ such that

$$\|\alpha p + \beta\| < p^{-\theta},$$

where $\theta = 1/5 - \varepsilon, \varepsilon > 0$ is arbitrary small. After that many authors improved the upper bound of the exponent $\theta$. The best result is given by Matomäki [10] with $\theta = 1/3 - \varepsilon$. Another interesting problem is the study of the distribution of the fractional part of $\alpha p^k$ with $2 \leq k \leq 12$, such Baker and Harman [1], Wong [1] etc. For $2 \leq k \leq 12$ the best result is due to Baker and Harman [1].

In [13] Todorova and Tolev considered the primes $p$ such that $\|\alpha p + \beta\| < p^{-\theta}$ and $p + 2 = P_r$ and prove existence of such primes with $\theta = 1/100$ and $r = 4$. Later Matomäki [10] and San Ying Shi [11] have shown that this actually holds when $p + 2 = P_2$ and $\theta = 1/1000$ and $\theta = 1.5/100$ respectively.

In [12] Shi and Wu proved existence of infinitely many primes $p$ such that $\|\alpha p^2 + \beta\| < p^{-\theta}$ and $p + 2 = P_4$ with $0 < \theta < 2/375$. In 2021 Xue, Li and Zhang [14] improved the result of Shi and Wu with $0 < \theta < 10/1561$.

In this paper we evaluate exponential sums over well-separated numbers and improve the results of Shi, Wu and Xue, Li and Zhang.

We will say that $d$ is a well-separable number of level $D \geq 1$ if for any $H, S \geq 1$ with $HS = D$, there are integers $h \leq H, s \leq S$ such that $d = hs$.

**Theorem 1.** Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies conditions

$$\left| \alpha - \frac{a}{q} \right| < \frac{1}{q^2}, \quad a \in \mathbb{Z}, \quad q \in \mathbb{N}, \quad (a, q) = 1, \quad q \geq 1,$$

$K$ and $D$ are defined by (8), $\lambda(d)$ are complex numbers defined for $d \leq D$,

$$\lambda(d) \ll \tau(d) \text{ and } \lambda(d) \neq 0 \text{ if } d \text{ is well-separable number of level } D,$$

$c(k) \ll 1$ are complex numbers, $0 < |k| \leq K$. Then for any arbitrary small $\varepsilon > 0$ and $b \in \mathbb{Z}$ for the sum

$$W(x) = \sum_{d \leq D} \lambda(d) \sum_{1 \leq |k| \leq K} c(k) \sum_{n \equiv b \pmod{d}} e((\alpha n^2 + \beta)k) \Lambda(n)$$

we have

$$W \ll x^\varepsilon \left( \frac{xK}{\Delta^{2/3}} + \frac{xK}{q^{1/3}} + \frac{x\Delta^{2/3}}{q^{1/3}} + x^{\frac{2}{15}} \Delta^{2/3} K + x^{\frac{1}{15}} K^{1/3} q^{1/3} + x^{\frac{1}{3}} \Delta^{1/2} K^{1/2} q^{1/3} \right).$$

**Remark 1.** It is obvious that the Theorem 1 is true if function $\lambda(d)$ is well-factorable.

**Lemma 1.** Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies conditions (2), sum $W(x)$ is defined by (4), $\lambda(d)$ are complex numbers defined for $d \leq D$ and satisfying (3) and (8), $c(k) \ll 1$ are 40
complex numbers $0 < |k| \leq K$. Then there exist a sequence
$$\{x_j\}_{j=1}^{\infty}, \lim_{j \to \infty} x_j = \infty,$$
such that
$$W(x_j) \ll x_j^{1-\omega}, \quad j = 1, 2, 3, \ldots$$
for any $\omega > 0$.

**Theorem 2.** Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies conditions (2), $\beta \in \mathbb{R}$ and let
$$0 < \theta < \frac{1}{1296} - \eta,$$
where $\eta$ is arbitrary small fixed number. Then there are infinitely many primes $p$ satisfying $p + 2 = P_2$ and such that
$$\|\alpha p^2 + \beta\| < p^{-\theta}.$$
Then
\[
\sum_{n \leq X} \min \left( \frac{XY}{n}, \frac{1}{\|\alpha n\|} \right) \ll XY \left( \frac{1}{q} + \frac{1}{Y} + \frac{q}{XY} \right) \log(2Xq).
\]

**Proof.** See Lemma 2.2 from [15], ch. 2, §2.1. \(\Box\)

**Lemma 5.** Let \(\mu, \zeta \in \mathbb{N}, \alpha \in \mathbb{R} \setminus \mathbb{Q}, \) and \(\alpha\) satisfy conditions (2). Then for every arbitrary small \(\varepsilon > 0\) the inequality
\[
\sum_{m \sim M} \tau_{\mu}(m) \sum_{j \sim J} \tau_{\zeta}(j) \min \left\{ \frac{x}{m^2j}, \frac{1}{\|\alpha m^2j\|} \right\} \ll x^\varepsilon \left( MJ + \frac{x}{M^{3/2}} + \frac{x}{Mq^{1/2}} + x^{1/2}q^{1/2} \right)
\]
is fulfilled.

**Proof.** See Lemma 8, [10]. \(\Box\)

**Lemma 6.** If \(d \mid P(z), \ z < D^{1/2}, \) \(\lambda^\pm\) are Rosser’s weights and either \(\lambda^+(d) \neq 0\) or \(\lambda^-(d) \neq 0\) then \(d\) is well-separated number.

**Proof.** See Lemma 12.16, [3] \(\Box\)

**Theorem 3.** Let \(2 \leq z \leq D^{1/2}\) and \(s = \frac{\log D}{\log z}\). If
\[
A_d = \frac{\omega(d)}{d} x + r(x, d) \quad \text{if} \quad \mu(d) \neq 0
\]
\[
\sum_{z_1 \leq p < z_2} \frac{\omega(p)}{p} = \log \left( \frac{\log z_2}{\log z_1} \right) + O\left( \frac{1}{\log z_1} \right), \quad z_2 > z_1 \geq 2
\]
where \(\omega(d)\) is a multiplicative function, \(0 < \omega(p) < p, \ x > 1\) is independent of \(d\). Then
\[
xV(z) \left( f(s) + O\left( \frac{1}{(\log D)^{1/3}} \right) \right) \leq S(A, z) \leq xV(z) \left( F(s) + O\left( \frac{1}{(\log D)^{1/3}} \right) \right),
\]
where \(d\) are well-separated numbers of level \(D\), \(f(s), F(s)\) are determined by the following differential-difference equations
\[
F(s) = \frac{2e^\gamma}{s}, \quad f(s) = 0 \quad \text{if} \quad 0 < s \leq 2
\]
\[
(sF(s))' = f(s-1), \quad (sf(s))' = F(s-1) \quad \text{if} \quad s > 2,
\]
where \(\gamma\) denote the Euler’s constant.

**4. Auxiliary results.**

**Lemma 7.** Let \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\) satisfied conditions (2), \(M, S, J, x \in \mathbb{R}^+, \ x > M^3S^2J\) and \(\mu, \sigma, \zeta \in [2, \infty) \cap \mathbb{N}, \)
\[
G = \sum_{m \sim M} \tau_{\mu}(m) \sum_{s \sim S} \tau_{\sigma}(s) \sum_{j \sim J} \tau_{\zeta}(j) \min \left\{ \frac{x}{m^3s^2j}, \frac{1}{\|\alpha m^3s^2j\|} \right\}.
\]
Then for any \(\varepsilon > 0\) the inequalities
\[
G \ll x^\varepsilon \left( MSJ + \frac{x}{M^4S} + \frac{x}{M^2S^2} + \frac{x}{M^2q^{1/2}} + \frac{x^{7/2}q^{1/2}}{M^2S} \right)
\]
and
\[ G \ll x^\varepsilon \left( MSJ + \frac{x}{M^4 S^4} + \frac{x}{M^2 S^4 q^4} + \frac{x^3 q^4}{M^2 S^4} \right) \]
are fulfilled.

**Proof.** Our proof is similar to proof of Lemma 8, [10]. Let
\[ H = \frac{x}{M^3 S^2 J}. \]
If \( H \leq 2 \), then trivially from Lemma 2 (iv) we get
\[ G \ll x^\varepsilon MSJ. \]
So we can assume that \( H > 2 \). From Lemma 2 (iv) it is obviously that
\[ G \ll x^\varepsilon \sum_{m \sim M} \sum_{s \sim S} \sum_{j \sim J} \min \left\{ \frac{x}{m^3 s^2 j^2}, \frac{1}{\|\alpha m s^2 j\|} \right\}. \]

We apply the Fourier expansion to the function \( \min \left\{ \frac{x}{m^3 s^2 j^2}, \frac{1}{\|\alpha m s^2 j\|} \right\} \) and get
\[ \min \left\{ \frac{x}{m^3 s^2 j^2}, \frac{1}{\|\alpha m s^2 j\|} \right\} = \sum_{0 < |h| \leq H^2} w(h)e(\alpha m s^2 jh) + O(\log x), \]
where
\[ w(h) \ll \min \left\{ \log H, \frac{H}{|h|} \right\}. \]
Then
\[ G \ll x^\varepsilon \sum_{0 < |h| \leq H^2} |w(h)| \sum_{s \sim S} \sum_{j \sim J} \left| \sum_{m \sim M} e(\alpha m s^2 jh) \right| + MSJ \log x. \]
So if
\[ G(H_0) = \sum_{h \sim H_0} \sum_{s \sim S} \sum_{j \sim J} \left| \sum_{m \sim M} e(\alpha m s^2 jh) \right|. \]
then using (13) we have
\[ G \ll x^\varepsilon \left( MSJ + \max_{1 \leq H_0 \leq H_1} G(H_0) + \max_{H_1 \leq H_0 \leq H^2} H \frac{H}{H_0} G(H_0) \right). \]
We shall evaluate the sum \( G(H_0) \). Applying the Cauchy–Schwarz inequality we obtain
\[ G^2(H_0) \ll x^\varepsilon H_0 JS \sum_{h \sim H_0} \sum_{s \sim S} \sum_{j \sim J} \left| \sum_{m \sim M} e(\alpha m s^2 jh) \right|^2 \ll x^\varepsilon H_0 JS \sum_{h \sim H_0} \sum_{s \sim S} \sum_{j \sim J} \sum_{m_1 \sim M} \sum_{m_2 \sim M} e(\alpha (m_1^3 - m_2^3) s^2 jh). \]
Substituting \( m_1 = m_2 + t \), where \( 0 \leq |t| \leq M \) we get
\[ G^2(H_0) \ll x^\varepsilon \left( H_0^2 J^2 S^2 M + H_0 J S G_1(H_0) \right), \]
We will estimate the above sum in two ways. Using Lemma 5 we obtain

\[ G_1(H_0) = \left| \sum_{h \sim H_0} \sum_{s \sim S} \sum_{j \sim J} \sum_{0 \lt |t| \lt M} \sum_{m_2 \sim M} e(\alpha(3m_2^2t + 3m_2t^2)s^2jh) \right|. \]

Applying again the Cauchy–Schwarz inequality we obtain

\[ G_1^2(H_0) \ll H_0JS^4 \sum_{h \sim H_0} \sum_{s \sim S} \sum_{j \sim J} \sum_{0 \lt |t| \lt M} \sum_{m_2 \sim M} e(\alpha(3m_2^2t + 3m_2t^2)s^2jh). \]

Choosing \( m_2 = m_3 + \ell \), where \( 0 \leq |\ell| \leq M \) we get

\[ G_1^2(H_0) \ll H_0^2J^2S^2M^3 + H_0JS^4 \sum_{u \sim H_0J^2M^2} \tau_5(u) \sum_{s \sim S} \sum_{m_3 \sim M} e(\alpha m_3\ell^2 s^2j). \]

Let \( u = 6t\ell h_0j \). Then using Lemma 3 and Lemma 5 we get

\[ G_1^2(H_0) \ll H_0^2J^2S^2M^3 + H_0JS^4 \sum_{u \sim H_0J^2M^2} \tau_5(u) \sum_{s \sim S} \sum_{m_3 \sim M} \min \left\{ \frac{H_0JS^2M^3}{s^2u}, \frac{1}{||\alpha s^2u||} \right\}. \]

We will estimate the above sum in two ways. Using Lemma 5 we obtain

\[ G_1^2(H_0) \ll x^\varepsilon \left( H_0^2J^2S^2M^3 + H_0^2J^2S^2M^4 + \frac{H_0^2J^2S^2M^4}{q^2} + H_0^2J^2S^2M^2q^2 \right). \]

So from (16)

\[ G(H_0) \ll x^\varepsilon \left( H_0JS^4 + H_0JS^2M + \frac{H_0JS^2M}{q^2} + H_0^2J^2S^2M^2q^2 \right). \]

Choosing \( H_0 = H \) from (11), (15) and (18) we get (9).

On the other hand we can write the inequality (17) as

\[ G_1^2(H_0) \ll H_0^2J^2S^2M^3 + H_0JS^4 \sum_{k \sim H_0JS^2M^2} \min \left\{ \frac{H_0JS^2M^3}{k}, \frac{1}{||\alpha k||} \right\} \]

and using Lemma 4 and (16) we get

\[ G^2(H_0) \ll x^\varepsilon \left( H_0^2J^2S^3M^3 + \frac{H_0^2J^2S^3M^4}{q} + H_0JS^2M \right). \]

Now we choose \( H_0 = H \). Then from (11), (15) and (19) the inequality (10) is received. \( \square \)

5. Proof of Theorem 1. To prove Theorem 1 we shall evaluate the sum \( W \) in two ways:

when \( x^{8/27} \Delta \leq D \leq \frac{x^{1/2}}{\Delta k^4} \), we will use the Vaughan’s identity;

when \( D \leq x^{8/27} \Delta \), we will use the Heat-Brown identity.
5.1. Evaluation by Vaughan’s identity. Let \( x^{8/27} \Delta \leq D \leq x^{1/2}/\Delta K^4 \) and \( 0 < |K| \leq \delta^{-1} \log^2 x \). First we decompose the sum \( W(x) \) into \( O(\log^2 x) \) sums of type

\[
W = W(x, D, K) = \sum_{d \sim D} \lambda(d) \sum_{1 \leq |k| \sim K} c(k) \sum_{n+2 \equiv 0(d)} e((\alpha n^2 + \beta)k) \Lambda(n),
\]

where \( \lambda(d) \) is Roser weight and in particular a necessary condition for \( \lambda(d) \neq 0 \) is numbers \( d \) are squarefree. So from this point on we will use that numbers \( d \) are squarefree. Then by Vaughan’s identity we can decompose the sum \( W \) into \( O(\log x) \) type I sums

\[
W_1 = \sum_{d \sim D \atop (a, d) = 1} \lambda(d) \sum_{k \sim K} c(k)e(\beta k) \sum_{m \sim M \atop \ell \sim L} a(m)e(\alpha(m\ell)^2k)
\]

or

\[
W_1' = \sum_{d \sim D \atop (a, d) = 1} \lambda(d) \sum_{k \sim K} c(k)e(\beta k) \sum_{m \sim M \atop \ell \sim L} \log(n)e(\alpha(m\ell)^2k)
\]

with \( M \leq x^{1/3} \) and \( O(\log x) \) type II sums

\[
W_2 = \sum_{d \sim D \atop (a, d) = 1} \lambda(d) \sum_{k \sim K} c(k)e(\beta k) \sum_{m \sim M \atop \ell \sim L} a(m)b(\ell)e(\alpha(m\ell)^2k)
\]

with \( M \in [x^{1/3}, x^{2/3}] \) and

\[
ML \sim x, \quad a(m) \ll \tau_3(m) \log m, \quad b(\ell) \ll \tau_3(\ell) \log \ell
\]

5.1.1. Evaluation of type II sums. The proof follows proof of Theorem 1, [10]. As \( x^{1/3} \leq M, L \leq x^{2/3} \) and \( ML \sim x \) we will consider only the case \( x^{1/2} \leq M \leq x^{2/3} \). The evaluation in the case \( x^{1/2} \leq L \leq x^{2/3} \) is the same. Using that \( d \) is well-separated numbers we write \( d = hs \), where \( (h, s) = 1 \) as \( d \) is squarefree. So the sum \( W_2 \) is presented as \( O(\log^2 x) \) sums of the type

\[
W_2 = \sum_{h \sim H \atop (h, a) = 1} \sum_{s \sim S \atop (s, ah) = 1} \lambda(hs) \sum_{k \sim K} c(k)e(\beta k) \sum_{\ell \sim L \atop m \sim M} a(m)b(\ell)e(\alpha(m\ell)^2k).
\]

Here

\[
h \sim H, \quad s \sim S, \quad D \sim HS
\]

and \( H \) we will choose later. Applying the Cauchy–Schwarz inequality to \( W_2 \) and using and Lemma 2(i) we obtain that
We apply again the Cauchy–Schwarz inequality and get
\[
W_2^2 \ll x^e K HM \sum_{k \sim K} \sum_{h \sim H} \sum_{s_1' \sim S} \lambda(h s_1') \sum_{s_2' \sim S} \lambda(h s_2')
\times \sum_{\ell_1 \sim L} b(\ell_1) \sum_{\ell_2 \sim L} b(\ell_2) \sum_{m \sim M} e(\alpha m^2 (\ell_1^2 - \ell_2^2)k).
\]

Let \((s_1', s_1') = r, s_1' = rs_1, s_2' = rs_2, r \sim R, R \leq S\) and \(s_1', s_2' \sim S/R\). Then
\[
W_2^2 \ll x^e K HM \sum_{k \sim K} \sum_{h \sim H} \sum_{r \sim R} \sum_{s_1' \sim S/R} \lambda(h s_1') \sum_{s_2' \sim S/R} \lambda(h s_2')
\times \sum_{\ell_1 \sim L} b(\ell_1) \sum_{\ell_2 \sim L} b(\ell_2) \sum_{m \sim M} e(\alpha m^2 (\ell_1^2 - \ell_2^2)k)
\]
\[
= W_{21} + W_{22},
\]
where \(W_{21}\) is this one part of above sum for which
\[
\ell_1 = \ell_2, \quad s_1 \neq s_2 \quad \text{or}
\]
\[
\ell_1 = \ell_2, \quad s_1 = s_2 = 1 \quad r \sim S \quad \text{or}
\]
\[
\ell_1 \neq \ell_2, \quad s_1 = s_2 = 1 \quad r \sim S \quad \text{or}
\]
\[
\ell_1 \neq \ell_2, \quad s_1 \neq s_2, \quad M < \frac{4HS^2}{R},
\]
\(W_{22}\) is the rest part of sum for \(W_2^2\). To evaluate the sum \(W_{21}\) we consider the cases \(x^{1/2} \leq M \leq \frac{x}{D}\) and \(x^{1/3} \leq L \leq D\). Then using Lemma 2 we get
\[
W_{21} \ll x^e \left( xMHK^2 + xD^2K^2 + \frac{x^2HK^2}{D} + \frac{xLD^2K^2}{H} \right).
\]
It is clear that for sum \(W_{22}\) we have \(\ell_1 \neq \ell_2, s_1 \neq s_2, M > \frac{4HS^2}{R}\). From \(m_1 \equiv a(hs_1), m_2 \equiv a(hs_2)\) follows that \(\ell_1 \equiv \ell_2(hr)\).

We apply again the Cauchy–Schwarz inequality and get
\[
W_{22}^2 \ll \frac{x^{2+\varepsilon}D^2K^3}{R^2} \sum_{k \sim K} \sum_{h \sim H} \sum_{r \sim R} \sum_{s_1' \sim S/R} \sum_{s_2' \sim S/R} \sum_{m_1 \sim M} \sum_{m_2 \sim M}
\times \sum_{\ell_1 \sim L} \sum_{\ell_2 \sim L} \sum_{m_1 \ell_1 \equiv a(hs_1)} \sum_{m_2 \ell_2 \equiv a(hs_2)} e(\alpha (m_1^2 - m_2^2) (\ell_1^2 - \ell_2^2)k).
\]
Let \(W_{221}\) be this one part of above sum for which \(m_1 = m_2\) and \(W_{222}\) be this part for
which $m_1 \neq m_2$. It is not difficult to see that

\[(25) \quad W_{221} \ll \frac{x^{3+\varepsilon}LD^2K^4}{H}.
\]

Let consider the sum $W_{222}$. As

\[m_i\ell_1 \equiv a \pmod{hrs_1} \quad \text{and} \quad m_i\ell_2 \equiv a \pmod{hrs_2}, \quad i = 1, 2\]

we get

\[m_1 \equiv m_2 (\pmod{hrs_1s_2}) \equiv f(m_1hrs_1s_2), \quad \text{where} \quad f = f(h, r, s_1, s_2, \ell_1, \ell_2)\]

and $\ell_1 \equiv \ell_2 (\pmod{hr})$. Let

\[m_1 = m_2 + hrs_1s_2t, \quad 0 < |t| \leq \frac{8MR}{HS^2} \quad \text{and} \quad \ell_1 = \ell_2 + hru, \quad 0 < |u| \leq \frac{2L}{HR}.
\]

Then

\[m_1^2 - m_2^2 = 2m_2hrs_1s_2t + h^2r^2s_1^2s_2^2t^2 \quad \text{and} \quad \ell_1^2 - \ell_2^2 = hru(2\ell_2 + hru).
\]

So using above equalities and Lemma 3 we obtain

\[W_{222} \ll \frac{x^{2+\varepsilon}D^2K^3}{R^2} \sum_{k \sim K} \sum_{h \sim H} \sum_{r \sim R} \sum_{s_1 \sim S/R} \sum_{s_2 \sim S/R} \sum_{(s_1, ah) = 1} \sum_{(s_2, ah) = 1} \min \left\{ \frac{1}{hrs_1s_2}, \frac{1}{\|2ahr^3s_1^2s_2^2tu\ell k\|} \right\},
\]

where $\ell = 2\ell_2 + hru$. We put

\[m = hr, \quad s = s_1s_2, \quad j = 2tunk, \quad j \ll \frac{xLK}{D^2}
\]

and it is clear that the sum $W_{222}$ can be represented as a finite number of sums of the type

\[W_{223} = \frac{x^{2+\varepsilon}D^2K^3}{R^2} \sum_{m \sim HR} \tau(m) \sum_{s \sim R^2/s^2} \tau(s) \sum_{j \ll \frac{xLK}{D^2}} \tau_5(j) \min \left\{ \frac{x^2K}{m^3s^2j}, \frac{1}{\|\alpha m^3s^2j\|} \right\}.
\]

Using Lemma 7, (21), (23), (24) and (25) we get

\[(26) \quad W_{223} \ll x^\varepsilon \left( x^{\frac{1}{2}}M^2H^2K + x^{\frac{1}{2}}DK + \frac{x^2L^2DK}{H^2} + \frac{x^3L^2DK}{H^2} + \frac{xH\frac{1}{2}K}{D^2}
\]

\[+ \frac{xK}{H^\frac{1}{2}} + \frac{xH\frac{1}{2}K}{D^\frac{1}{2}} + \frac{xK}{q^{1/2}} + \frac{x^2K^{3/2}q^{1/2}}{\Delta K^4} \right)
\]

According to $D, M$ and $L$ we have

\[(27) \quad W_{223} \ll x^\varepsilon \left( V_1 + V_2 + V_3 + V_4 \right),
\]

where $V_1$ is the sum with

\[(28) \quad x^{1/2} \leq M \leq \frac{x}{D}, \quad x^{2/5} \leq D \leq \frac{x^{1/2}}{\Delta K^4},
\]

\[D \leq L \leq x^{1/2},
\]
$V_2$ is the sum with

$$\frac{x}{D} \leq M < x^{1/3} D^{2/3}, \quad x^{2/5} \leq D \leq \frac{x^{1/2}}{\Delta K^4}$$

(29)

and $V_3$ is the sum with

$$x^{1/3} D^{2/3} \leq M \leq x^{2/3}, \quad x^{2/5} \leq D \leq \frac{x^{1/2}}{\Delta K^4}$$

(30)

and $V_4$ is the sum with

$$x^{1/2} \leq M \leq x^{2/3}, \quad x^{8/27} \Delta \leq D \leq x^{2/5}$$

(31)

For sums $V_1$, $V_2$, $V_3$ and $V_4$ we choose consequently

$$H = \frac{D}{\Delta^{1/2}}, \quad H = \frac{LD^{2/3}}{x^{1/3}}, \quad H = \frac{x^{1/3}}{\Delta}, \quad H = \frac{L^{4/5} D^{9/5}}{x^{4/5}}.$$

and from (26), (28), (29), (30) and (31) we get

$$W_2 \ll \begin{cases} 
  x^e \left( \frac{xK}{\Delta^{1/2}} + x^{5/6} D^{1/2} \Delta^{1/2} K + \frac{xK}{q^{1/2}} + x^{15/27} K^{31/32} q^{1/32} \right), & \text{if } x^{2/5} \leq D \leq \frac{x^{1/2}}{\Delta K^4}, \\
  x^e \left( \frac{x^{31/32} K}{D^{1/2}} + \frac{xK}{q^{1/2}} + x^{15/27} K^{31/32} q^{1/32} \right), & \text{if } x^{8/27} \Delta \leq D \leq x^{2/5}.
\end{cases}$$

So

$$W_2 \ll x^e \left( \frac{xK}{\Delta^{1/2}} + x^{5/6} D^{1/2} \Delta^{1/2} K + \frac{xK}{q^{1/2}} + x^{15/27} K^{31/32} q^{1/32} \right).$$

(32)

5.1.2. Evaluation of type I sums. In this case we have that $L > x^{2/3}$ and $M < x^{3/4}$. Again we will use that $d$ is well-separated numbers. So we can write $d = hs$ with $h$ and $s$ satisfying conditions (21) and we will choose $H$ later. So the sum $W_1$ is presented as $O(\log^2 x)$ sums of the type

$$W_1 = \sum_{h \sim H} \sum_{s \sim S} \lambda(hs) \sum_{k \sim K} c(k)e(\beta k) \sum_{\ell \sim L} \sum_{m \sim M} a(m)e(\alpha(m\ell)^2 k).$$

Working in the same way as in the evaluation of the sum $W_2$ see (22), we get

$$W_1^2 \ll x^e KHM \sum_{k \sim K} \sum_{h \sim H} \sum_{r \sim R} \sum_{s_1 \sim S/R} \lambda(hrs_1) \sum_{s_2 \sim S/R} \lambda(hrs_2)$$

$$\times \sum_{\ell_1 \sim \Lambda} b(\ell_1) \sum_{\ell_2 \sim \Lambda} b(\ell_2) \sum_{m \sim M} \sum_{m \sim M} e(\alpha m^2 (\ell_1^2 - \ell_2^2) k)$$

$$= W_{11} + W_{12} + W_{13},$$

48
where $W_{12}$ is this one part of above sum for which
\[ \ell_1 = \ell_2, \]
$W_{13}$ is this one part of above sum for which
\[ \ell_1 \neq \ell_2, \quad s_1 = s_2 = 1 \quad r \sim S \]
and $W_{11}$ is the rest part of sum for $W^2$. Using that $L > x^{\frac{2}{3}}$ and $M < x^{\frac{1}{5}}$ we get

\[
(34) \quad W_{12} \ll x^\varepsilon x M H K^2.
\]

For the sum $W_{13}$ we get

\[
W_{13} \ll x^\varepsilon KHM \sum_{k \sim K} \sum_{d \sim D} \sum_{m \sim M} \left| \sum_{\ell_i \sim L, m \ell_i \equiv a(d)}^{t \sim T} e(\alpha m^2 (\ell_1^2 - \ell_2^2)k) \right|.
\]

As $\ell_1 \equiv \ell_2 (\text{mod } d)$ we put

\[
\ell_1 = \ell_2 + du, \quad 0 < |u| \ll \frac{L}{D}.
\]

So

\[
W_{13} \ll x^\varepsilon KHM \sum_{k \sim K} \sum_{d \sim D} \sum_{m \sim M} \sum_{u \ll \frac{L}{D}} \left| \sum_{\ell_2 \sim L, m \ell_2 \equiv a(d)}^{t \sim T} e(2\alpha m^2 \ell_2 udk) \right|
\]

and from Lemma 3 we get

\[
W_{13} \ll x^\varepsilon KHM \sum_{k \sim K} \sum_{d \sim D} \sum_{m \sim M} \sum_{u \ll \frac{L}{D}} \min \left\{ \frac{x^2 K}{m^2 d^2 (2uk)}, \frac{1}{\|\alpha m^2 d^2 (2uk)\|} \right\}
\]

The above sum can be represented as a finite number of sums of the type

\[
W_{14} \ll x^\varepsilon KHM \sum_{z \sim MD} \tau(z) \sum_{t \ll \frac{L}{D}} \tau_3(t) \min \left\{ \frac{x^2 K}{z^2 |t^2 \| \|\alpha z^2 t\|} \right\}
\]

Using Lemma 5 and $ML \sim x$ we obtain

\[
(35) \quad W_{13}^{\frac{1}{2}} \ll x^{\varepsilon} \left( x^{\frac{13}{2}} M^{\frac{1}{2}} H^\frac{1}{2} K + \frac{x^{\frac{13}{2}} L^\frac{1}{2} H^\frac{1}{2} K}{D^\frac{1}{2}} + \frac{x H^\frac{1}{2} K}{D^\frac{3}{2} q^\frac{1}{2}} + \frac{x^{\frac{13}{2}} H^\frac{1}{2} q^\frac{1}{2} K^\frac{1}{2}}{D^\frac{3}{2}} \right)
\]

Using analogous reasoning for the sum $W_{11}$ we get

\[
(36) \quad W_{11} \ll x^\varepsilon KHM \sum_{k \sim K} \sum_{h \sim H} \sum_{r \sim R} \sum_{s_i \sim S/R} \sum_{s_1 s_2, ah = 1} \left| \sum_{m \sim M} \sum_{u \ll \frac{L}{D}} \right| \sum_{\ell_2 \sim L, m \ell_2 \equiv a(hrs_2)}^{m \ell_2 + uhr \equiv a(s_1)} e(2\alpha m^2 \ell_2 uhrk) \right|
\]
and from Lemma 3 we obtain

\[ W_{11} \ll x^\varepsilon KHM \sum_{k \sim K} \sum_{h \sim H} \sum_{r \sim R} \sum_{s_i \sim S/R} \sum_{i=1,2} \sum_{m \sim M} \sum_{u \ll \frac{1}{\pi^2 \gamma}} \min \left\{ \frac{L}{hfs_1s_2}, \| \alpha m^2 h^2 r^2 \| s_1 s_2 u k \right\}. \]

Applying Lemma 5 and using that \( ML \sim x \) we get

\[ W_{11}^{1/2} \ll x^\varepsilon \left( \frac{x \bar{K}}{H} + \frac{x^2 L \bar{K}}{H^2} + \frac{xK}{q} + x^2 q^2 K \right). \]

Choosing \( H = \frac{D}{M^2} \) from (34), (35), (37), (22) and (5) follows

\[ W_1 \ll x^\varepsilon \left( \frac{xK}{D^4} + \frac{xK}{q^4} + x^2 K \right). \]

From (32) and (38) it follows that in the case \( x^{8/27} \Delta \leq D \leq x^{1/2} \Delta K^4 \) the estimate

\[ W \ll x^\varepsilon \left( \frac{xK}{\Delta^4} + x^\frac{5}{7} D^\frac{1}{4} \Delta^\frac{1}{4} K + xK + x^\frac{15}{16} K^3 \right) \]

is fulfilled.

5.2. Evaluation by Heat-Brown’s identity. Let \( D \leq x^{8/27} \Delta \). We decompose the sum \( W(x) \) into \( O(\log^2 x) \) as in (4). Using Heath-Brown’s identity [6] with parameters

\[ P = x/2, P_1 = x, u = \frac{x^{1/3}}{2^{21} D}, v = 2^7 x^{1/3}, w = 2^7 x^{1/3} D^\frac{1}{4}, \]

we decompose the sum \( W \) as a linear combination of \( O(\log^6 N) \) sums of first and second type. The sums of the first type are

\[ W_1 = \sum_{d \leq D} \lambda(d) \sum_{0 < |k| \leq K} c(k)e(\beta k) \sum_{M < m \leq M_1} a_m \sum_{L < \ell \leq L_1 \atop m \ell = -2(d)} e(\alpha m^2 \ell^2 k) \]

and

\[ W'_1 = \sum_{d \leq D} \lambda(d) \sum_{0 < |k| \leq K} c(k)e(\beta k) \sum_{M < m \leq M_1} a(m) \sum_{L < \ell \leq L_1 \atop m \ell = -2(d)} \log \ell e(\alpha m^2 \ell^2 k), \]

where

\[ M_1 \leq 2M, \quad L_1 \leq 2L, \quad ML \asymp x, \quad L \geq w, \quad a(m) \ll x^\varepsilon. \]

The sums of the second type are

\[ W_2 = \sum_{d \leq D} \lambda(d) \sum_{0 < |k| \leq K} c(k)e(\beta k) \sum_{L < \ell \leq L_1 \atop m \ell = -2(d)} b(\ell) \sum_{M < m \leq M_1} a(m)e(\alpha m^2 \ell^2 k), \]

where

\[ M_1 \leq 2M, \quad L_1 \leq 2L, \quad ML \asymp x, \quad u \leq L \leq v, \quad a(m), b(\ell) \ll x^\varepsilon. \]
5.2.1. Evaluation of type II sums. Applying the Cauchy–Schwarz inequality to $W_2$ and using Lemma 2(i), (40), (42) and (46) we obtain that

$$W_2^2 \ll x^6 KDM \sum_{k \sim K} \sum_{d \sim D} \sum_{\ell_1, \ell_2} b(\ell_1)b(\ell_2) \sum_{m_i \sim M} \sum_{\ell_1 \equiv \ell_2 (d)} e(\alpha m^2 (\ell_1^2 - \ell_2^2)k)$$

$$= W_{21} + x^{1+\varepsilon} MDK^2$$

where

$$W_{21} = x^6 KDM \sum_{k \sim K} \sum_{d \sim D} \sum_{\ell_1, \ell_2} b(\ell_1)b(\ell_2) \sum_{m_i \sim M} \sum_{\ell_1 \equiv \ell_2 (d)} e(\alpha m^2 (\ell_1^2 - \ell_2^2)k)$$

Applying again the Cauchy–Schwarz inequality and substituting

$$m_1 = m_2 + td, \ t \ll \frac{M}{D} \text{ and } \ell_1 = \ell_2 + \omega d, \ \omega \ll \frac{L}{D}$$

we sequentially obtain

$$W_{21}^2 \ll x^{2+\varepsilon} D^2K^3 \sum_{k \sim K} \sum_{d \sim D} \sum_{\ell_1, \ell_2} \sum_{m_i \sim M} \sum_{\ell_1 \equiv \ell_2 (d)} e(\alpha (m_1^2 - m_2^2)(\ell_1^2 - \ell_2^2)k)$$

$$+ x^{3+\varepsilon} LDK^4$$

$$= W_{22} + x^{3+\varepsilon} LDK^4$$

with

$$W_{22} \ll x^{2+\varepsilon} D^2K^3 \sum_{k \sim K} \sum_{d \sim D} \sum_{\omega \ll \frac{L}{k}, \ t \ll \frac{M}{d}} \min \left\{ \frac{M}{d}, \left\| \alpha d^3 \omega (2\ell_2 + \omega d)k \right\| \right\}$$

Putting $\ell = \ell_2 + \omega d$ and $z = \omega \ell k t$ we get

$$W_{22} \ll x^{2+\varepsilon} D^2K^3 \sum_{d \sim D} \sum_{z \ll \frac{LK}{d^2}} \tau_4(z) \min \left\{ \frac{x^2 K}{d^3 z}, \frac{1}{\left\| \alpha d^3 z \right\|} \right\}$$

If

$$\Delta < D \leq x^{\frac{3}{7}} \Delta$$

then from inequality (10) of Lemma 7, (43), (44) and (45) we get

$$W_2 \ll x^\varepsilon \left( x^{\frac{77}{49}} \Delta \frac{q^2}{\sqrt{d}} K + \frac{xK}{q^{\frac{1}{4}}} + \frac{xK}{\Delta^{\frac{1}{8}}} + x^{\frac{15}{16}} q^3 K^4 \right)$$

If

$$D \leq \Delta$$

we will estimate the sum $W_{22}$ by putting $u = d^3 z$ and applying Lemma 4, Lemma 2 (ii) to find

$$W_2 \ll x^\varepsilon \left( x^{\frac{77}{49}} \Delta \frac{q^2}{\sqrt{d}} K + \frac{x\Delta^\frac{1}{2}}{q^{\frac{1}{4}}} + \frac{xK}{q^{\frac{1}{8}}} + \frac{xK}{\Delta^{\frac{1}{8}}} + x^{\frac{15}{16}} q^3 K^4 \right).$$
From (47) and (50) we get

\[(50) \quad W_2 \ll x^\varepsilon \left( \frac{x}{2} \Delta^\frac{3}{2} K + \frac{x}{q^2} \Delta^\frac{1}{2} + \frac{x}{q^4} \Delta^\frac{1}{4} + \frac{x}{q^6} \Delta^\frac{1}{6} K + x^\frac{1}{4} \Delta^\frac{1}{4} K^\frac{3}{4} q^\frac{1}{2} \right) \]

5.2.2. Evaluation of type I sums. Reasoning as in the estimation of the sum \(W_{13}\) (see (5.1.2)) we obtain

\[(51) \quad W_1^2 \ll x^\varepsilon MDK \sum_{z \sim MD} \tau(z) \sum_{t \ll \frac{L}{K}} \tau_5(t) \min \left\{ \frac{x^2 K}{z^2 t}, \frac{1}{\|\alpha z^2 t\|} \right\} + x^\varepsilon MDK^2.\]

Using Lemma 5 and \(ML \sim x\) we obtain

\[(52) \quad W_1^2 \ll x^\varepsilon \left( MDK^2 + \frac{x^2 K^2}{(MD)^\frac{3}{2}} + \frac{x^2 K^2}{q^2} + \frac{xK^2}{q^2} + xK^\frac{3}{2} q^\frac{1}{2} \right) \]

Using the inequality (51) we will evaluate the sum \(W_1\) in one more way. Let \(u = z^2 t\) and from Lemma 4 (iii) and Lemma 2 (iv) we find

\[(53) \quad W_1^2 \ll x^\varepsilon MDK \sum_{u \ll xMDK} \tau_5(u) \min \left\{ \frac{x^2 K}{u}, \frac{1}{\|\alpha u\|} \right\} \ll x^\varepsilon \left( MDKq + xM^2 D^2 K^2 + \frac{x^2 MDK^2}{q} \right) \]

If \(MD > \Delta\) using the estimate (52) we get

\[(54) \quad W_1 \ll x^\varepsilon \left( x^{\frac{27}{45}} \Delta^{\frac{6}{10}} K + \frac{xK}{\Delta^\frac{1}{4}} + \frac{x}{q^\frac{1}{2}} + x^\frac{1}{2} K^\frac{3}{2} q^\frac{1}{2} \right). \]

If \(MD \leq \Delta\) from (53) it follows

\[(55) \quad W_1 \ll x^\varepsilon \left( x^{\frac{1}{2}} \Delta K + \frac{x}{q^\frac{1}{2}} \Delta^\frac{1}{2} K^\frac{1}{2} q^\frac{1}{2} \right), \]

then using (54) and (55) we get

\[(56) \quad W_1 \ll x^\varepsilon \left( x^{\frac{27}{45}} \Delta^{\frac{6}{10}} K + \frac{xK}{\Delta^\frac{1}{4}} + \frac{x}{q^\frac{1}{2}} + x^\frac{1}{2} K^\frac{3}{2} q^\frac{1}{2} \right). \]

From (39), (50) and (56)

\[(57) \quad W \ll x^\varepsilon \left( \frac{xK}{\Delta^\frac{1}{2}} + \frac{xK}{q^\frac{1}{2}} + x^\frac{1}{2} \Delta^\frac{1}{2} K^\frac{3}{4} q^\frac{1}{2} \right). \]

5.3. Proof of Lemma 1. In Theorem 2 choose

\[(58) \quad x = q, \quad \Delta = K^{\frac{22}{34}}, \quad K = x^\frac{1}{20m}, \quad D = \frac{x^{1/2}}{\Delta K^{1/4}}, \]

where \(\eta\) is arbitrary small fixed number.

6. Proof of Theorem 2. As in [13] we take a periodic function with period 1 such that

\[(59) \quad 0 < \chi(t) < 1 \quad \text{if} \quad -\delta < t < \delta; \]

\[\chi(t) = 0 \quad \text{if} \quad \delta \leq t \leq 1 - \delta, \]

52
and which has a Fourier series

$$\chi(t) = \delta + \sum_{|k| > 0} c(k)e(kt),$$

with coefficients satisfying

$$c(0) = \delta,$$

$$c(k) \ll \delta \quad \text{for all } k,$$

$$\sum_{|k| > K} |c(k)| \ll x^{-1}$$

and $\delta$ and $K$ satisfying the conditions (8).

The existence of such a function is a consequence of a well known lemma of Vinogradov (see [9], ch. 1, §2).

Next we will use sieve methods. As usual, for any sequence $A$ of integers weighted by the numbers $f_n$, $n \in A$, we set

$$S(A, z) = \sum_{n \in A, (n, P(z)) = 1} f_n$$

and denote by $A_d$ be the subsequence of elements $n \in A$ with $n \equiv 0 \pmod{d}$. We write

$$P(z) = \prod_{p < z} p, \quad V(z) = \prod_{p | P(z)} \left(1 - \frac{\omega(p)}{p}\right) \quad \text{and} \quad C_0 = \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right).$$

and we will use the linear sieve due to Iwaniec – this is Theorem 3 (see [7]).

To prove Theorem 2, it suffice to show that

$$S(A, N^{1/3}) = \sum_{n+2 \leq x, (n+2, P(x^{1/3})) = 1} \chi(\alpha n^2 + \beta) \Lambda(n) > 0.$$  

Following the exposition in Shi’s article (see [11]) we have that

$$S \geq \sum_{n+2 \leq x, (n+2, P(x^{1/12})) = 1} \chi(\alpha n^2 + \beta) \Lambda(n) \left(1 - \frac{1}{2} \sum_{x^{1/12} < p_1 < x^{1/3}} \sum_{n \equiv -2(p_1)} 1 - \frac{1}{2} \sum_{n+2 = p_1 p_2 p_3} \right.$$

$$\left. \sum_{x^{1/12} < p_1 < x^{1/3}} \sum_{x^{1/3} < p_2 < \left(\frac{x}{p_1}\right)^{1/2}} 1 + O(x^{11/12}) \right).$$

So
\[ S \geq S(A, x^{1/12}) - \frac{1}{2} \sum_{x^{1/12} \leq p_1 < x^{1/3.1}} S(A_{p_1}, x^{1/12}) - \frac{1}{2} \sum_{x^{1/3.1} \leq p_1 < x^{1/3.1} \leq p_2 < \left( \frac{x}{p_1} \right)^{1/2}} S(A_{p_1p_2}, x^{1/12}) \]

\[ - \sum_{x^{1/12} \leq p_1 < p_2 < \left( \frac{x}{p_1} \right)^{1/2}} S(A_{p_1p_2}, x^{1/12}) + O(x^{11/12}) \]

\[ = S_1 - \frac{1}{2} S_2 - \frac{1}{2} S_3 - S_4 + O(x^{11/12}) \]

and it is enough to proof that above expression is positive. Consider a square-free number \( d \). If \( 2 \mid d \), then we write \( |A_d| = |r(A, d)| \leq 1 \). Otherwise we have by the Fourier expansion of \( \chi(n) \) that

\[ |A_d| = \sum_{\frac{n}{\phi(d)} \equiv -1(d)} \chi(\alpha n^2 + \beta) \Lambda(n) \]

\[ = \sum_{\frac{n}{\phi(d)} \equiv -1(d)} \left( \delta + \delta \sum_{0 < |k| < K} c(k) e(\alpha n^2 k) \Lambda(n) + O(x^{-1}) \right) \]

\[ = \delta \left( \frac{x}{\phi(d)} + R_1(d) + R_2(d) + O\left( \frac{x}{d(\log x)^4} \right) \right). \]

Here \( c(k) \ll 1 \),

\[ R_1(d) = \sum_{\frac{p}{\phi(d)} \equiv -1(d)} \frac{1 - x}{\phi(d)} \]

\[ R_2(d) = \sum_{0 < |k| < K} c(k) \sum_{\frac{n}{\phi(d)} \equiv -2(d)} e(\alpha n^2 k) \Lambda(n) \]

Applying Bombieri–Vinogradov theorem (see [8], Theorem 17.1)

\[ \sum_{d \leq D} |R_1(d)| \ll \frac{x}{(\log x)^4}. \]

On the other hand, Theorem 1 implies that for a well-separated numbers \( d \) of level \( D = \frac{x^{1/2}}{\Delta K^4} \) and \( \lambda(d) \ll \tau(d) \) we get

\[ \sum_{d \leq D} \lambda(d) R_2(d) \ll \frac{x}{(\log x)^4} \]

when \( q = x \), where \( a/q \) convergent to \( \alpha \) with a large enough denominator. From here on, the reasoning we go through is the same as in Shi’s paper (see [11]). We will only note
that to estimate the sum
\[ \sum_{x^{1/12} \leq p < x^{1/3}} \sum_{d \leq D} R_2(pd) \]
with \( D = \frac{x^{1/2}}{\Delta p K^4} \) first we present it as a \( O(\log^4 x) \) number of sums of the type
\[ R_2(P) = \sum_{d \sim D} \lambda(d) \sum_{1 \leq |k| \sim K} c(k) \sum_{p \sim P} \sum_{n \sim x \atop n+2 \equiv 0 (p_1)} e(\alpha n^2 k) \Lambda(n), \]
where \( x^{1/12} \leq P < x^{1/3}/2. \)

If \( DP \leq x^{8/27} \Delta \) we put \( t = dp \) and represent the sum \( R_2(P) \) in type:
\[ R_2(P) = \sum_{1 \leq |k| \sim K} c(k) \sum_{t \sim DP} g(t, d) \sum_{n \sim x \atop n+2 \equiv 0 (t)} e(\alpha n^2 k) \Lambda(n), \]
where
\[ g(t, d) = \sum_{d \sim D \atop d(t, P(z)) \atop t/d > x^{1/12}} \lambda(d) \ll \tau(t) \]
and evaluation is in the same way as in §5.2.

If \( DP \geq x^{8/27} \Delta \) then, depending on which interval \( P \) falls into, and bearing in mind Remark 2 and the fact that \( d \) is well-separated, we choose \( H \) so that \( PH \) falls into one of the intervals \( x^{2/5} \leq PH \leq x^{1/2} \frac{1}{\Delta K^4} \) or \( x^{8/27} \Delta \leq PH \leq x^{2/5} \). So
\[ R_2(P) = \sum_{1 \leq |k| \sim K} c(k) \sum_{s \sim S} \sum_{h \sim H} \lambda(hs) \sum_{p_1 \sim P} \sum_{n \sim x \atop n+2 \equiv 0 (d) \atop n+2 \equiv 0 (p_1)} e(\alpha n^2 k) \Lambda(n) \]
\[ = \sum_{1 \leq |k| \sim K} c(k) \sum_{s \sim S} \sum_{t \sim PH} g(t, s) \sum_{n \sim x \atop n+2 \equiv 0 (ts)} e(\alpha n^2 k) \Lambda(n) \]
where
\[ g(t, s) = \sum_{h \sim H \atop h|t, P(z)} \lambda(hs) \ll \tau(t) \]
and evaluation is in the same way as in §5.1.

Using the same calculation as in [11] with
\[ z = x^{1/12}, \quad \Delta = K^{33/34}, \quad K = x^{1/1200 - \eta}, \quad D = \frac{x^{1/2}}{\Delta K^4}, \]
we get that inequality (62) is true and the proof of Theorem is complete.
REFERENCES